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1987 J. Phys. A: Math. Gen. 20 L815

(http://iopscience.iop.org/0305-4470/20/13/002)

J. Phys. A: Math. Gen. 20 (1987) L815–L819. Printed in the UK

LETTER TO THE EDITOR

A possible generalisation of the coherent states for free fields

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Received 26 May 1987

Abstract. It is shown explicitly how a Bogoliubov type transformation acting on coherent states gives rise to states where the mean values of the field strengths keep the original values while the mean values of the squared strengths or the correlation functions can, within some bounds, be prescribed independently.

The coherent states are, for systems having harmonic oscillator like dynamics, quantum states that approach the classical configurations in a simple way, because they are simply related to the mean value of the observables (Glauber 1970, Klauder and Skagerstam 1985). It is, however, evident that this simple and strict relation with the mean value could even be too strict in the sense that once the mean value is given the dispersion is also given. As a trivial example let us take the one-dimensional oscillator and consider a coherent state such that $\langle c|x|c\rangle = \xi$ and $\langle c|p|c\rangle = \eta$, then one gets $\langle c|x^2|c\rangle = \xi^2 + \frac{1}{2}$, $\langle c|p^2|c\rangle = \eta^2 + \frac{1}{2}$.

There are many possibilities of modifying the definition of the state $|c\rangle$ in such a way as to keep the former pair of relations while alternating the latter pair; some of them appear very natural because they are the most straightforward generalisation of the transformations that generate the coherent states.

In this letter one of these generalisations is worked out, the system under consideration being the electromagnetic field, because for this system the coherent states are of particular interest and significance.

Let us start with some notation and conventions.

The free EM field is described through a vector potential in Coulomb gauge, at a fixed time $t = 0$,

$$A_i(\boldsymbol{x}) = (2\pi)^{-3/2} \int a_i(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\, \mathrm{d}^3k$$

$$= (2\pi)^{-3/2} \sum_{l=\pm} [c^l(\boldsymbol{k}) e_i^l(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) + \mathrm{HC}]\, \mathrm{d}^3k/2\omega \tag{1a}$$

$$\dot{A}_i(\boldsymbol{x}) = (2\pi)^{-3/2} \int \dot{a}_i(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\, \mathrm{d}^3k$$

$$= (2\pi)^{-3/2} \sum_{l=\pm} [-\mathrm{i}c^l(\boldsymbol{k}) e_i^l(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) + \mathrm{HC}]\, \mathrm{d}^3k \tag{1b}$$

$\boldsymbol{e}^l(k)$ are the polarisation vectors with the condition $\boldsymbol{k}\cdot\boldsymbol{e} = 0$; $\omega = |k|$. The commutation relations are

$$[c^l(\boldsymbol{k}),\ c^{l'}(\boldsymbol{k}')] = \delta_{ll'} 2\omega \delta^3(\boldsymbol{k} - \boldsymbol{k}') \tag{2}$$

0305-4470/87/130815+05$02.50

which give

$$[a_i(\boldsymbol{k}),\ \dot{a}_j(\boldsymbol{k}')] = \mathrm{i}\eta_{ij}\delta^3(\boldsymbol{k}+\boldsymbol{k}') \qquad \text{with } \eta_{ij} = \delta_{ij} - k_i k_j/\omega^2. \tag{3}$$

The definition of the coherent states is given through the construction of the operator $\mathcal{U} = \mathrm{e}^T$

$$T = \int [c^l(\boldsymbol{k}) f^{l*}(\boldsymbol{k}) - c^{l\dagger}(\boldsymbol{k}) f^l(\boldsymbol{k})]\, \mathrm{d}^3k/2\omega \tag{4}$$

having the properties

$$\begin{aligned} \mathcal{U} c^l(\boldsymbol{k}) \mathcal{U}^\dagger &= c^l(\boldsymbol{k}) + f^l(\boldsymbol{k}) \\ \mathcal{U} c^{l\dagger}(\boldsymbol{k}) \mathcal{U}^\dagger &= c^{l\dagger}(\boldsymbol{k}) + f^{l*}(\boldsymbol{k}). \end{aligned} \tag{5}$$

In fact, in this way the state $|f\rangle\!\rangle = \mathcal{U}^\dagger|\rangle$ has the properties that the mean value of the electric field is (in Fourier components)

$$\langle\!\langle f|(-\dot{a}_i(\boldsymbol{k}))|f\rangle\!\rangle = \varepsilon_i(\boldsymbol{k}) \tag{6a}$$

with

$$\varepsilon_i(\boldsymbol{k}) = \tfrac{1}{2}\mathrm{i} \sum_{l=\pm} [f^l(\boldsymbol{k}) e_i^l(\boldsymbol{k}) - f^{l*}(-\boldsymbol{k}) e_i^l(-\boldsymbol{k})]$$

and the mean value of the magnetic field is

$$\langle\!\langle f|(\mathrm{i}\varepsilon_{imn} k_m a_n(\boldsymbol{k}))|f\rangle\!\rangle = \beta_i(\boldsymbol{k}) \tag{6b}$$

with

$$\beta_i(\boldsymbol{k}) = \tfrac{1}{2}\mathrm{i} \sum_{l=\pm} \varepsilon_{imn}(k_m/\omega)[f^l(\boldsymbol{k}) e_i^l(\boldsymbol{k}) + f^{l*}(-\boldsymbol{k}) e_i^{l*}(-\boldsymbol{k})].$$

As stated at the beginning, the mean value of the squared field strengths is obviously strictly defined, for instance

$$\langle\!\langle f|\dot{a}_i(\boldsymbol{k})\dot{a}_j(\boldsymbol{q})|f\rangle\!\rangle = \tfrac{1}{2}\omega\eta_{ij}(\boldsymbol{k})\delta^3(\boldsymbol{k}+\boldsymbol{q}) + \varepsilon_i(\boldsymbol{k})\varepsilon_j(\boldsymbol{k}) \tag{7}$$

and the δ function is the consequence of the singularity in x space.

The simplest possibility of changing the result of (7) while keeping the results of (6) stems from the observation that the transformation induced by $\mathcal{U}$ is linear and inhomogeneous and one could try a linear homogeneous transformation of the type of a bosonic Bogoliubov-Valatin transformation (Bogoliubov *et al* 1958, Valatin 1958). Since the transformation $\mathcal{U}$ is effectively acting on a particular mode of the field $a(\boldsymbol{k})$ the same can be done for the homogeneous case by defining $\mathcal{V} = \mathrm{e}^{\mathrm{i}G}$,

$$G = \tfrac{1}{2}\Lambda \int a_i(\boldsymbol{k}') u_i(-\boldsymbol{k}')\, \mathrm{d}^3k' \int \dot{a}_j(\boldsymbol{k}) v_j(-\boldsymbol{k})\, \mathrm{d}^3k/2\omega + \text{HC} \tag{8}$$

with real Λ and the reality condition $u_i^*(\boldsymbol{k}) = u_i(-\boldsymbol{k})$, $v_i^*(\boldsymbol{k}) = v_i(-\boldsymbol{k})$.

The Coulomb gauge implies $\boldsymbol{k}\cdot\boldsymbol{u} = \boldsymbol{k}\cdot\boldsymbol{v} = 0$ and the presence of the free parameter Λ allows the normalisation condition

$$\int u_i(\boldsymbol{k}) v_i(-\boldsymbol{k})\, \mathrm{d}^3k/2\omega = 1.$$

The analogues of equation (5) are

$$\mathcal{V} a_i(\boldsymbol{k})\mathcal{V}^\dagger = a_i(\boldsymbol{k}) + (\mathrm{e}^{\Lambda}-1)\frac{1}{2\omega}\, v_i(\boldsymbol{k}) \int a_j(\boldsymbol{k}')u_j(-\boldsymbol{k}')\, \mathrm{d}^3k' \tag{9a}$$

$$\mathcal{V} \dot{a}_i(\boldsymbol{k})\mathcal{V}^\dagger = \dot{a}_i(\boldsymbol{k}) + (\mathrm{e}^{-\Lambda}-1)u_i(\boldsymbol{k}) \int \dot{a}_j(\boldsymbol{k}')v_j(-\boldsymbol{k}')\, \mathrm{d}^3k'/2\omega' \tag{9b}$$

and the homogeneity of the transformation, together with the obvious fact that $\langle|a(\boldsymbol{k})|\rangle = \langle|\dot{a}(\boldsymbol{k})|\rangle = 0$ is enough to yield the result that, defining $|f;\, uv\rangle\!\rangle = \mathcal{U}^\dagger\mathcal{V}^\dagger|\rangle$, one obtains

$$\langle\!\langle f;\, uv|a|f;\, uv\rangle\!\rangle \equiv \langle\!\langle f|a|f\rangle\!\rangle \tag{10}$$

with a corresponding statement for $\dot{a}$, while the expectation values of the quadratic expressions are modified.

In order to discuss results more explicitly it is better to get rid of the singularity (in x space). To this end we can define the smeared field operators† (Bohr and Rosenfeld 1933, Ferretti 1954)

$$\boldsymbol{E}_W = -\int \dot{\boldsymbol{A}}(\boldsymbol{x})\, W(\boldsymbol{x})\, \mathrm{d}^3x \qquad \boldsymbol{B}_W = \int \operatorname{curl} \boldsymbol{A}(\boldsymbol{x})\, W(\boldsymbol{x})\, \mathrm{d}^3x$$

with $\int W(\boldsymbol{x})\, \mathrm{d}^3x = 1$.

In the following, with a certain loss of generality but with a simplification in the formulae we shall consider, in equation (8), the particular case $\boldsymbol{u} = \boldsymbol{v}$.

Using the notation previously introduced, denoting by $w(\boldsymbol{k})$ the Fourier transform $W(\boldsymbol{x})$ we obtain

$$\langle\!\langle f;\, v|\boldsymbol{E}_W|f;\, v\rangle\!\rangle = \int \boldsymbol{\varepsilon}(\boldsymbol{k})w(-\boldsymbol{k})\, \mathrm{d}^3k \tag{11a}$$

$$\langle\!\langle f;\, v|\boldsymbol{B}_W|f;\, v\rangle\!\rangle = \int \boldsymbol{\beta}(\boldsymbol{k})w(-\boldsymbol{k})\, \mathrm{d}^3k \tag{11b}$$

$$\langle\!\langle f;\, v|E_W^2|f;\, v\rangle\!\rangle = \langle|E_W^2|\rangle + \left(\int \boldsymbol{\varepsilon}(\boldsymbol{k})w(-\boldsymbol{k})\, \mathrm{d}^3k\right)^2 + \tfrac{1}{4}(\mathrm{e}^{-2\Lambda}-1)\left(\int \boldsymbol{v}(\boldsymbol{k})w(-\boldsymbol{k})\, \mathrm{d}^3k\right)^2 \tag{12a}$$

$$\langle\!\langle f;\, v|B_W^2|f;\, v\rangle\!\rangle = \langle|B_W^2|\rangle + \left(\int \boldsymbol{\beta}(\boldsymbol{k})w(-\boldsymbol{k})\, \mathrm{d}^3k\right)^2 + \tfrac{1}{4}(\mathrm{e}^{2\Lambda}-1)\left(\int \boldsymbol{v}(\boldsymbol{k})w(-\boldsymbol{k})\, \mathrm{d}^3k\right)^2. \tag{12b}$$

Since Λ can take both signs it appears that the mean value of E_W^2, for instance, can be changed in both directions, either making it bigger or smaller, the mean value of B_W^2 changing always in the opposite direction.

The region over which the field strengths are averaged is quite arbitrary, provided the edges are not too sharp, a form like $W \propto \vartheta(l-|\boldsymbol{x}|)$ is not acceptable (Rosenfeld 1955), choosing a standard Gaussian shape $W \propto \exp(-x^2/2S)$ the result is

$$\langle|E_W^2|\rangle = \langle|B_W^2|\rangle = (2\pi S)^{-2}.$$

† Since we work at fixed time the smearing is performed only over the space.

The fact that the mean value of E^2 and B^2 move in opposite directions does not yield a complete compensation. This is evident considering the total energy

$$\mathcal{H}=\tfrac{1}{2}\int :(E^2+B^2):\,\mathrm{d}^3x$$

we have

$$\langle\langle f;\,v|\mathcal{H}|f;\,v\rangle\rangle=\tfrac{1}{2}\int \boldsymbol{\varepsilon}(\boldsymbol{k})\cdot\boldsymbol{\varepsilon}(-\boldsymbol{k})\,\mathrm{d}^3k+\tfrac{1}{2}\int \boldsymbol{\beta}(\boldsymbol{k})\cdot\boldsymbol{\beta}(-\boldsymbol{k})\,\mathrm{d}^3k$$
$$+\tfrac{1}{4}(\cosh\Lambda-1)\int \boldsymbol{v}(\boldsymbol{k})\cdot\boldsymbol{v}(-\boldsymbol{k})\,\mathrm{d}^3k$$

so that the energy, at fixed mean values of the fields, $\boldsymbol{\varepsilon}(\boldsymbol{k})$, $\boldsymbol{\beta}(\boldsymbol{k})$ takes the smallest value for $\Lambda=0$, i.e. for a pure coherent state. There is still a formal problem that must be settled: the transformations $\mathcal{U}$ are clearly commutative while the transformation $\mathcal{V}$ are not; they are commutative only if the functions v are orthogonal in the sense that for $\mathcal{V}_1$ and $\mathcal{V}_2$

$$\int \boldsymbol{v}^{(1)}(\boldsymbol{k})\cdot\boldsymbol{v}^{(2)}(-\boldsymbol{k})\,\mathrm{d}^3k/2\omega=0.$$

The mutual behaviour can be symbolically written $\mathcal{V}\mathcal{U}=\mathcal{U}'\mathcal{V}$ where $\mathcal{U}'$ is again a transformation of type $\mathcal{U}$, with different f. Explicitly, given f and v, we have

$$f^{l\prime}(\boldsymbol{q})=f^l(\boldsymbol{q})+\boldsymbol{e}^{l*}(\boldsymbol{q})\cdot\boldsymbol{v}(\boldsymbol{q})\bigg((\cosh\Lambda-1)\int \boldsymbol{v}(\boldsymbol{k})\cdot\boldsymbol{e}^l(-\boldsymbol{k})f^l(-\boldsymbol{k})\,\mathrm{d}^3k/2\omega$$
$$-\sinh\Lambda\int \boldsymbol{v}(\boldsymbol{k})\cdot\boldsymbol{e}^{l*}(\boldsymbol{k})f^{l*}(\boldsymbol{k})\,\mathrm{d}^3k/2\omega\bigg).$$

This expression shows that the order of the operators in defining the generalised coherent state is not very relevant, the one chosen in the presentation giving a simpler relation between the mean values.

In conclusion, a very simple generalisation of the coherent states for free fields has been presented, obtained through a particular form of the Bogoliubov transformation. A more general form of the transformation gives clearly a more general set of states, all of them keeping the property expressed by (11) which derives from the homogeneity of the transformation, since the standard coherent states depend only on a complex function of a single vector. The explicit derivation for this generalisation has been shown only for a transformation depending again on a function of a single vector, without taking into account the more general possibility that exists, of having

$$c^l(\boldsymbol{k})\to\int F^{lm}(\boldsymbol{k},\boldsymbol{q})c^m(\boldsymbol{q})\,\mathrm{d}^3q$$

where F is a suitable function of two vectors such that the transformation preserves the fundamental commutation relations.

The construction shows in a very transparent way how much larger the set of quantum states is with respect to the classical configurations. In fact, once the operator $\mathcal{U}$ has been chosen, the mean values are defined throughout all space, but other observables, in the examples bilinear in the field, acquire different mean values whenever we change the operator $\mathcal{V}$ at fixed $\mathcal{U}$. This is clearly also true for other kinds of bilinear forms, for instance correlations between fields at different points or between fields smeared over different regions of space.

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